TOPICAL REVIEW

Singular responses of spin-incoherent Luttinger liquids

Gregory A Fiete

Department of Physics, University of Texas at Austin, Austin, TX 78712, USA

E-mail: fiete@physics.utexas.edu

Abstract

When a local potential changes abruptly in time, an electron gas responds by shifting to a new state which at long times is orthogonal to the one in the absence of the local potential. This is known as Anderson's orthogonality catastrophe and it is relevant for the so-called x-ray edge or Fermi-edge singularity, and for tunneling into an interacting one-dimensional system of fermions. It often happens that the finite frequency response of the photon absorption or the tunneling density of states exhibits a singular behavior as a function of frequency: $(\frac{\omega_{th}}{\omega-\omega_{th}})^\alpha \Theta(\omega - \omega_{th})$, where ω_{th} is a threshold frequency and α is an exponent characterizing the singular response. In this review singular responses of spin-incoherent Luttinger liquids are reviewed. Such responses most often do not fall into the familiar form above, but instead typically exhibit logarithmic corrections and display a much higher universality in terms of the microscopic interactions in the theory. Specific predictions are made, the current experimental situation is summarized and key outstanding theoretical issues related to spin-incoherent Luttinger liquids are highlighted.

Contents

1. Introduction

Until a few years ago there was a common feeling in the condensed matter physics community that we had a very good understanding of the universal properties of one-dimensional systems of fermions through our knowledge of Luttinger liquid theory and its instabilities (Giamarchi 2004, Gogolin *et al* 1998). One of the key results of Luttinger liquid theory (Haldane 1981) is that interacting electrons in one dimension have separate dynamics in the spin and charge sectors at low energies. These low energy collective spin and charge modes are bosonic and generally propagate with different velocities, with the spin mode typically being the slower of the two. Luttinger liquid theory provides a particularly convenient theoretical framework for discussing the response of the electron gas to a time-dependent local potential, even in higher dimensions: the mathematical trick of 'bosonization' makes the evaluation of correlation functions relatively simple, and a potential local on the scale of the Fermi wavelength results in scattering of primarily s-wave type which effectively converts the problem to just one dimension (the radial coordinate).

A few years back it was realized that at finite temperatures a 'Luttinger liquid system' can take on qualitatively new behaviors in transport (Matveev 2004a) and the single-particle Green's function (Cheianov and Zvonarev 2004a). Some of the
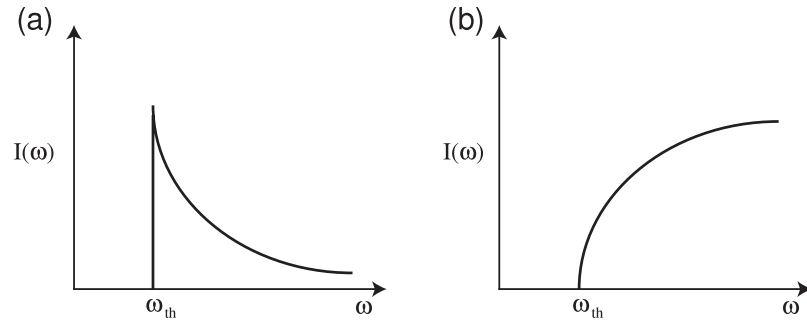
Figure 1. Schematic behavior of photon absorption intensity, equation (1), for an infinite mass core hole in three dimensions. The exponent α given by equation (2) contains positive contributions from excitonic effects and negative contributions from Anderson's orthogonality catastrophe. It is possible for α to be either positive (a) or negative (b). In the spin-incoherent regime of one-dimensional systems the absorption is not generally given by equation (1) but instead exhibits logarithmic corrections and additional physics enters the exponent α.

results for the single-particle Green's function were anticipated in an earlier work (Berkovich 1991), but the many novel features that can arise due to finite temperatures were not fully appreciated until the more recent works mentioned above which really started proper the study of the spin-incoherent Luttinger liquid (Fiete 2007).

The spin-incoherent regime of a one-dimensional interacting electron gas is defined in the following way. Let us assume that at zero temperature our system is spin–charge separated with spin velocity v_s and charge velocity v_c such that $v_s \ll v_c$. If the interactions are strong (regardless of the range of interactions) it can be shown that v_s can be *exponentially small* compared to v_c (Häusler 1996, Matveev 2004b, Fogler and Pivovarov 2005). Defining a characteristic spin energy $E_s = \hbar v_s/a$ and a characteristic charge energy $E_c = \hbar v_c/a$ for mean particle spacing a, it is evidently clear that the spin energy can be exponentially suppressed relative to the charge energy: $E_s \ll E_c$. An exponentially separated spin and charge energy makes it possible for the thermal energy to lie in the range $E_s \ll k_B T \ll E_c$, where k_B is Boltzmann's constant and T is the temperature. A one-dimensional electron system in the energy window $E_s \ll k_B T \ll E_c$ is said to be in the spin-incoherent regime. If the charge sector is gapless it will be described by a zero-temperature Luttinger liquid theory and the resulting system is called a spin-incoherent Luttinger liquid (SILL). Note that, if E_s refers to a spin *gap*, then an SILL is still realized when $E_s \ll k_B T \ll E_c$, provided the charge sector remains gapless. For a general discussion of the SILL see (Fiete 2007). For SILL physics in ferromagnetic and superconducting hybrid structures see (Tilahun and Fiete 2008, 2009), and in cold atomic gases see (Kakashvili *et al* 2008).

2. Familiar singular responses to a local time-dependent potential

In this review we will only discuss a particularly simple form of a time-dependent spatially local potential whereby its value is changed instantaneously from zero to non-zero. The prototypical physical realization of this situation is when a photon in the x-ray energy range ejects a deep 'core' electron from a metal—instantly changing the local charge by $+e$. The electrons of the metal then rush in to screen the positive

potential and thereby change their states. If the electrons interact strongly with each other the processes of the electrons 'rushing in' to screen the local potential will obviously be impacted. According to Luttinger liquid theory, interactions qualitatively change the nature of the low energy physics in such a way that electron-like quasi-particles are absent from the theory (Voit 1995), leaving only the collective bosonic excitations. On this basis alone, we expect the response of one-dimensional systems to differ in some ways from their higher-dimensional counterparts. In fact, this expectation is correct but to fully appreciate the differences, it is important to review some of basic ideas and qualitative expectations for the singular responses of non-interacting electron systems.

2.1. Singular response of a non-interacting Fermi sea

The seminal theoretical ideas of the singular response of a non-interacting electron gas due to a localized core hole were worked out by (Mahan 1967) and (Nozières and de Dominicis 1969). A nice textbook-level discussion can be found in (Mahan 1981) and a more recent review in (Ohtaka and Tanabe 1990). The most important results to emerge are: (i) there is a threshold frequency ω_{th} above which there is photon absorption and below which there is not. (ii) The photon absorption at frequencies just above ω_{th} follows a power law with exponent α. The photon absorption intensity is computed from a Fermi golden-rule-type treatment in the electron–photon coupling and from the results just cited above takes the form

$$I(\omega) = A_0 \left(\frac{\omega_{th}}{\omega - \omega_{th}} \right)^{\alpha} \Theta(\omega - \omega_{th}), \qquad (1)$$

near threshold where A_0 is a constant, and $\Theta(x) = 0$ for $x < 0$ while $\Theta(x) = 1$ for $x > 0$. A schematic of the possible behaviors is shown in figure 1. The important microscopic details of the electron dynamics and the electron–core hole interaction are encoded in the exponent α and the power law form itself. The power law arises from summing an infinite series of diagrams containing logarithmic divergences (Mahan 1967). As we will see, in the SILL the power law form is not sacred. It will be modified by logarithmic corrections, and the physics encoded in α will depend on whether or not the one-dimensional system is in the spin-incoherent regime.

For a non-interacting electron gas the exponent takes the form (Nozières and de Dominicis 1969)

$$\alpha = \frac{2\delta_0(k_\mathrm{F})}{\pi} - 2\sum_l (2l+1)\left[\frac{\delta_l(k_\mathrm{F})}{\pi}\right]^2, \qquad (2)$$

where $\delta_l(k_\mathrm{F})$ is the phase shift induced by the core hole in the lth channel of the conduction electrons. Often s-wave scattering ($l = 0$) is the dominant channel and we will assume this is the case throughout this review. As the total amount of screening charge around the core hole must be one unit of charge, there is a Friedel sum rule:

$$1 = \frac{2}{\pi}\sum_l (2l+1)\delta_l(k_\mathrm{F}). \qquad (3)$$

If the scattering is not too strong, the phase shifts $\delta_l(k_\mathrm{F})$ can be computed in the Born approximation and one finds (Mahan 1981) $\delta_0(k_\mathrm{F})/\pi = N(E_\mathrm{F})V_0$, where $N(E_\mathrm{F})$ is the density of states at the Fermi energy and $V_0 > 0$ is the strength of the electron–hole interaction assumed momentum- and energy-independent. (The actual core hole–electron interaction is $-V_0$ and therefore attractive.) Within the Born approximation and assuming only s-wave scattering, the exponent becomes

$$\alpha \approx 2N(E_\mathrm{F})V_0 - 2[N(E_\mathrm{F})V_0]^2. \qquad (4)$$

Evidently, equations (2) and (4) imply there are competing effects in α. The first term proportional to $\delta_0(k_\mathrm{F})$ is positive while the second term proportional to $[\delta_0(k_\mathrm{F})]^2$ is negative. Going back to equation (1) one sees that a positive contribution to α tends to make the threshold response more singular, while a negative contribution to α tends to make the threshold response less singular. In fact, the origin of these two contributions is very well understood. The first contribution to α in equation (2) comes from excitonic effects arising from the attractive interaction between the electrons and the core hole. The second contribution to α in equation (2) arises from Anderson's orthogonality catastrophe (Anderson 1967) and it tends to suppress the absorption by the small (vanishing as $\omega \to 0$) overlap between initial and final states. Note that for a non-interacting Fermi sea both excitonic effects and orthogonality effects disappear if $V_0 = 0$ and the absorption threshold becomes a simple step. Looking ahead for a moment to Luttinger liquid physics, it is instructive to point out that, if a collection of fermions is interacting, there will be orthogonality effects if an additional fermion of like charge is added to the system because the other fermions must adjust their state to accommodate the newcomer, but no excitonic effects. This rearrangement of electronic states is what is responsible for the famous power law suppression of the tunneling density of states in a Luttinger liquid (Giamarchi 2004, Gogolin et al 1998). Finally, it is worth emphasizing that, depending on the microscopic details involved, it is possible for α to be positive or negative as shown in figure 1. If the mass of the core hole is finite and one considers only direct transitions like those in figure 2 one finds that the threshold is smeared out and disappears (Müller-Hartmann et al 1971, Ruckenstein and Schmitt-Rink 1987). We will find this is not the case when we go to one dimension.
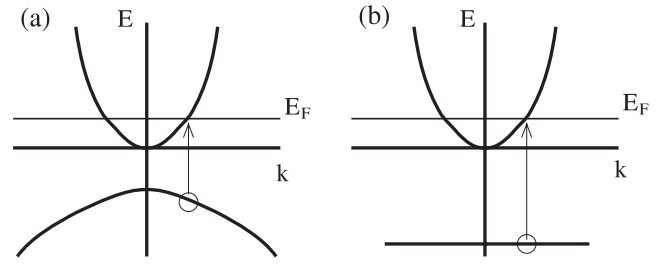

Figure 2. Schematic of the possible direct transitions for exciting a hole with (a) finite mass and (b) infinite mass. Here E_F denotes the Fermi energy. In (a) an electron is excited from a valence band up to a conduction band, leaving behind a single mobile hole in the valence band. In (b) an electron is excited from an infinite mass core level. The photo-absorption of both process (a) and (b) differ in one dimension compared to their counterparts in three dimensions. Furthermore, the response in one dimension depends on whether the system is in the Luttinger liquid or spin-incoherent Luttinger liquid regime.

2.2. Singular response of a Luttinger liquid

Before we turn our attention to the spin-incoherent Luttinger liquid, it is useful for orientation to discuss the singular response of its closest cousin, the Luttinger liquid. The behavior exhibited by the one-dimensional Luttinger liquid (LL) will already take us part of the way to understanding the response of the SILL. In all of the results quoted below finite system size effects will not become important until frequencies smaller than v_F/L are probed, where L is the length of the system and v_F is the Fermi velocity. Also, finite temperature will cut off the singular behaviors when relevant frequencies are of order $k_\mathrm{B}T$ and smaller. In this sense, finite system size and temperature can act as a non-zero threshold for tunneling into a finite length system or an infinite system at finite temperature, for example.

2.2.1. Tunneling into a Luttinger liquid.
Let us first consider the simplest kind of singular response: abruptly adding a particle to an LL as occurs during tunneling. A nice pedagogical discussion of the essential physics of tunneling into an LL is given by (Fisher and Glazman 1997). Here we adopt the central elements of their discussion. The tunneling process is conveniently visualized within a Wigner crystal type picture for spinless electrons, as shown in figure 3. Such a picture implies very strong electron interactions, but the physical result and the final mathematical expression that describes it are actually valid at any interaction strength. We imagine a process where an electron tunnels from the left of the barrier at $x = 0$ to the right of the barrier at time $t = 0$. The rate at which the electron tunnels is determined by the Debye frequency of the Wigner crystal, $\omega_\mathrm{D} = k_\mathrm{F}v$, where k_F is the Fermi wavevector and v is the phonon (charge) velocity. When an electron tunnels, the potential near the barrier suddenly (on a timescale $\sim 1/\omega_\mathrm{D}$) changes. The energy cost of this new configuration is large, $\sim \omega_\mathrm{D}/g$, where $g \ll 1$ is the interaction parameter for the spinless Luttinger liquid. (Here $g = 1$ if the electrons are non-interacting.) This large energy cost is reduced by the relatively slow process of the electrons relaxing
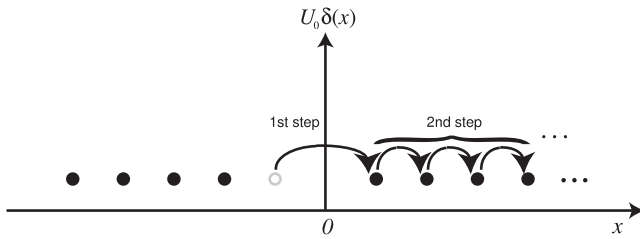
Figure 3. Schematic of the tunneling process of spinless electrons
into a 'Wigner crystal' type electron arrangement. Electrons are
schematically represented by black dots evenly spaced on either side
of the tunneling barrier of strength U_0 at $x = 0$. The tunneling occurs
in two steps: a fast process where the electron enters the region
$x > 0$ from the left followed by a slow process of charge relaxation
where each electron shifts over to the right by a distance of one
lattice constant. At long times $\langle \psi(\tau) \psi^\dagger(0) \rangle \sim (\frac{1}{\omega_D \tau})^{1/g}$ as discussed
in the text. The resulting behavior of the tunneling density of states is
$A(\omega) \sim \omega^{1/g-1}$. The qualitative behavior of the tunneling density of
states looks very much like that in figure 1(b) with $\omega_{th} = 0$.

to new equilibrium positions in order to minimize the energy.
Once the relaxation process is complete, all the electrons
will be shifted to the right by one crystalline period. In the
thermodynamic limit this new wavefunction is orthogonal to
the original one which implies at zero energy ($\omega = 0$) there
is no tunneling. This is a concrete example of Anderson's
orthogonality catastrophe that we earlier mentioned plays a
role in the exponent α of the threshold behavior of photon
absorption. For finite energy the final state may be in an excited
state so the relaxation is not complete and the initial and final
wavefunction can have finite overlap. We will now see how
this directly leads to a power law tunneling density of states,
one of the 'singular' responses of interest in this review.

We can compute the tunneling density of states from
the single-particle Green's function (Mahan 1981). In the
semiclassical approximation the electron tunneling takes the
path of minimal action which is given by the classical trajectory

$$\langle \psi(x, t) \psi^\dagger(x, 0) \rangle \sim \exp[-S(t)], \qquad (5)$$

where $\psi(x, t)$ is the electron annihilation operator at position
x and time t, and $\psi^\dagger(x, t)$ is the creation operator. The initial
density deformation $\rho(x, t = 0) = 2\delta(x)$ will eventually relax
to $\rho(x, t = \infty) = 0$. As the process is classically forbidden,
the trajectory runs in imaginary time $t = i\tau$. The classical
equation of motions for a harmonic lattice give

$$\rho(x, \tau) = \frac{2}{\pi} \frac{v\tau}{x^2 + (v\tau)^2}, \qquad x > 0. \qquad (6)$$

The potential energy of the deformation is

$$V_{def}(\tau) = \frac{v}{g} \frac{\pi}{2} \int_0^\infty dx \, \rho(x, \tau)^2 = \frac{2v}{\pi g} \int_0^\infty dx \, \frac{(v\tau)^2}{[x^2 + (v\tau)^2]^2}, \qquad (7)$$

which gives $V_{def}(\tau) = 1/(2g\tau)$. By the virial theorem, the
kinetic energy is equal to the potential energy. Therefore the
action describing the slow part of the tunneling is

$$S(\tau) = 2 \int_{\omega_D^{-1}}^{\tau} d\tau' \, V_{def}(\tau') = \frac{1}{g} \ln(\omega_D \tau), \qquad (8)$$

whence $\langle \psi(x, t) \psi^\dagger(x, 0) \rangle \sim 1/(\omega_D \tau)^{1/g}$. Taking the Fourier
transform one finds the tunneling density of states $A(\omega) \sim
\omega^{1/g-1}$. Since we have assumed $g < 1$, the tunneling density
of states $\to 0$ as $\omega \to 0$, as we noted above in the context of
the orthogonality catastrophe.

The discussion above illustrates some of the most central
aspects of singular responses of interacting one-dimensional
systems. However, there is much more to the story. Let us
now suppose that, instead of an electron tunneling, we consider
an incident photon that abruptly creates a particle–hole pair
as shown in figure 2. Note that, due to the presence of the
gap between the hole states and conduction band, there will be
a minimum frequency ω_{th} required to create the particle–hole
pair. If we first focus on the electrons in the conduction band,
we see that the process is similar to tunneling: conduction
band electrons see the sudden appearance of a new electron
and they must adjust their positions by relaxing to a new state
just as if the electron tunneled from outside the system. As
we have seen, this relaxation rate is related to the strength of
the electron interactions g in the Luttinger liquid and, based
on the power law density of states we obtained, we expect the
threshold behavior will also contain at least a power law
contribution with an exponent that contains information about
the interactions in the system.

It is also important to consider the interaction between
the hole created and the electrons in the conduction band. As
holes and electrons generally have different masses and they
attract rather than repel each other, the creation of a hole adds
new physics to the electron tunneling physics above. In fact,
to compute the photon absorption spectrum near threshold we
must *simultaneously* account for the interaction of the electrons
with each other and with the hole. In general, this is a very
difficult problem. Fortunately, though, the power of Luttinger
liquid theory and the commonly used technical methods such
as bosonization allow us to make some precise and quite
general statements about $I(\omega)$.

2.2.2. Fermi-edge singularity in a Luttinger liquid.
Pioneering theoretical work in the study of the Fermi-edge
singularity in Luttinger liquids was carried out more than
15 years ago (Ogawa *et al* 1992, Lee and Chen 1992), and
later works soon followed (Kane *et al* 1994, Affleck and
Ludwig 1994, Otani and Ogawa 1996a, 1996b, Furusaki
1997, Kominik *et al* 1997, Tsukamoto *et al* 1998a, 1998b,
Balents 2000). A recent review of the optical response
of low-dimensional systems (including some aspects of one
dimension) exists (Ogawa 2004), so here we will only highlight
the main issues and results to set the stage for our discussion
of the spin-incoherent one-dimensional systems.

In one-dimensional electronic systems the effect of im-
purities is especially dramatic compared to three dimensions.
Even for an unrealistic non-interacting one-dimensional sys-
tem the electrons localize, leading to insulating behavior. How-
ever, for a realistic system which contains repulsive interac-
tions even a single local potential of arbitrary strength will lo-
calize the electrons to the half-line (Kane and Fisher 1992, Fu-
rusaki and Nagaosa 1993). The central physics of this result
is the relevance of electron backscattering as a perturbation: at

low energies the strength of the impurity grows to arbitrarily large values and eventually divides the system into two semi-infinite segments. This physics is important for the Fermi-edge singularity involving an infinitely massive core hole, i.e. one that is localized in space.

On the other hand, when the mass of the core hole is finite the physics is very different. The properties of a heavy particle (holes are typically heavier than electrons) moving in a Luttinger liquid have been investigated (Castro Neto and Fisher 1996). It was shown that the mobility of the heavy particle diverges at low energies, implying that electron backscattering from it becomes irrelevant. We therefore see that the Fermi-edge singularity in a Luttinger liquid falls into two categories: (i) infinite hole mass and (ii) finite hole mass. Note that in three dimensions it was also important to divide the physics into the same two categories, but the reason for that division was different. In three dimensions finite hole mass led to excitonic effects that smeared the edge singularity of the direct transition and eliminated the threshold. This has to do with a large (compared to one dimension) phase space for recoil effects. A sharp threshold only exists for the infinitely massive hole. On the other hand, in one dimension the issue is about the relevance/irrelevance of backscattering and that is decided by the interactions (repulsive or attractive) in the LL. Another way to see that it is natural to divide the classes into infinite and finite masses cases is that the symmetry is different. An infinitely massive hole breaks the translational symmetry of the problem, while a hole with finite mass does not.

One of the first issues addressed regarding the Fermi-edge singularity in the LL is whether the threshold still exists for the case of finite hole mass. This was answered in the affirmative (Ogawa *et al* 1992). The next logical question is how does the threshold exponent α depend on the electron–hole and electron–electron interactions? Initially it was argued that α was independent of the hole mass (Ogawa *et al* 1992), but this was later challenged and corrected (Castella 1996, Tsukamoto *et al* 1998a, 1998b). Knowing that any finite mass hole is an irrelevant perturbation (Castro Neto and Fisher 1996) we can immediately see that only forward scattering interactions will contribute to α. Indeed this is found as well as a schematic phase diagram for when α is positive or negative (Ogawa *et al* 1992). The result is (Tsukamoto *et al* 1998a)

$$\alpha_{\text{mobile}}^{\text{LL}} = 1 - \frac{1}{2}\left[g(1-\delta_{\text{a}})^2 + \frac{1}{g}(1-\delta_{\text{s}})^2\right], \qquad (9)$$

where δ_{a} is an asymmetric phase shift and δ_{s} a symmetric phase shift defined below in the discussion of the spin-incoherent case. The presence of δ_{a} implies that the exponent depends on the mass of the hole. Note that both δ_{a} and δ_{s} correspond to forward scattering processes and $g < 1$ is the Luttinger parameter for repulsively interacting spinless electrons. We can directly compare the result, equation (9), with the non-interacting case, equation (2), by taking $g = 1$ and $\delta_{\text{a}} = 0$ (appropriate for an infinitely massive hole). In doing so, we note the form is identical to an s-wave-only scattering model. From this comparison we also see that δ_{s} has the meaning of a phase shift up to factors of 2 and π. Note that we are allowed to compare the result for a mobile impurity in an LL with

an infinitely massive impurity in a free electron gas in three dimensions because in the latter backscattering is not a relevant perturbation. They are therefore in a similar 'class'.

The situation is different when we turn to immobile infinite mass holes. Here, as before, a threshold exists, but now α is modified by the relevant backscattering and in fact obtains a universal contribution of 1/8 (Kane *et al* 1994, Prokof'ev 1994, Affleck and Ludwig 1994, Furusaki 1997, Kominik *et al* 1997):

$$\alpha_{\text{immobile}}^{\text{LL}} = 1 - \frac{1}{g}(1-\delta_{\text{s}})^2 - 1/8. \qquad (10)$$

In section 3 we will discuss how these two results are modified in the spin-incoherent regime. But before we do that, it is important to comment on the experimental situation. Unfortunately, there are few published results (Calleja *et al* 1991, Fritze *et al* 1993, Calleja *et al* 1995, Ihara *et al* 2007) and the overall level of agreement between theory and experiment is poor. Clearly there is a great need for further experiments on Fermi-edge singularities on clean, single-mode one-dimensional systems. We hope this review will help to stimulate more.

3. Tunneling and edge singularities in a spin-incoherent Luttinger liquid

We finally come to the main topic of this review well prepared from the earlier discussions of singular responses of non-interacting electron gases and Luttinger liquids. The question to ask is: what are the similarities and differences between the singular responses of an SILL and an LL or Fermi liquid? As we will see, the behavior is qualitatively different from both the LL and Fermi liquid, making it possible to distinguish them in experiment. Perhaps the simplest and most striking difference is the behavior of the tunneling density of states. We begin there.

3.1. Tunneling into a spin-incoherent Luttinger liquid

The first theoretical study of tunneling into an SILL was reported in the infinite U limit of the Hubbard model in a lengthy Bethe ansatz calculation (Cheianov and Zvonarev 2004a, 2004b). Immediately afterward a simpler and more general method based on bosonization and the imaginary time path integral representation was given (Fiete and Balents 2004). Extensions of the method to finite length systems including boundary effects and externally applied magnetic fields have also been reported (Fiete *et al* 2005b, Kindermann and Brouwer 2006, Kakashvili and Johannesson 2007, Fiete 2007). While the imaginary time path integral formulation is well suited to the regime of the SILL, it does not provide a conceptually straightforward generalization to the regime $k_{\text{B}}T \lesssim E_{\text{s}} \ll E_{\text{c}}$. In order to address that regime, we need a representation of the spin degrees of freedom that is valid at arbitrary energy scales. Such a formulation is now in hand (Matveev *et al* 2007a, 2007b) and it provides a good starting point for numerical studies in the regime $k_{\text{B}}T \lesssim E_{\text{s}} \ll E_{\text{c}}$.

3.1.1. Bosonization of strongly interacting electrons. As is typical of interacting one-dimensional electron systems, we assume the Hamiltonian in the strongly interacting case is spin–charge-separated (Matveev 2004b, Fiete 2007): $H = H_c + H_s$. Here, H_c is identical to that of the LL and is given by (Giamarchi 2004, Gogolin *et al* 1998)

$$H_c = v_c \int \frac{\mathrm{d}x}{2\pi} \left[\frac{1}{g}(\partial_x \theta(x))^2 + g(\partial_x \phi(x))^2 \right], \qquad (11)$$

where θ and ϕ are bosonic fields satisfying $[\theta(x), \phi(x')] = -\mathrm{i}\frac{\pi}{2}\mathrm{sgn}(x - x')$ and $g = 2g_c$ is the effective Luttinger interaction parameter of the electron gas (Fiete *et al* 2005a).

On the other hand, the spin Hamiltonian at arbitrary temperatures is, to a very good approximation, given by a nearest-neighbor antiferromagnetic Heisenberg spin chain (Matveev 2004b, Klironomos *et al* 2005, Fogler and Pivovarov 2005):

$$H_s = \sum_l J \vec{S}_l \cdot \vec{S}_{l+1}, \qquad (12)$$

where evidently the spin energy is set by J: $E_s = J$. The basic idea is to represent the electron operator as a product of operators that describe the holons $\Psi(x)$ (spinless fermions that naturally arise in the context of strongly interacting fermions and the spin-incoherent regime (Fiete *et al* 2005a, Fiete and Balents 2004)) and the spin degrees of freedom $\vec{S}_l$. The holon operators (denoted by $\Psi^\dagger$, Ψ) by construction satisfy the equation

$$\Psi^\dagger(x)\Psi(x) = \psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x), \qquad (13)$$

where ψ_s is the electron annihilation operator for electrons of spin projection σ and $\psi_\sigma^\dagger$ is the corresponding electron creation operator.

The issue of how to bosonize the electron operator for a strongly interacting system earlier arose in the context of the large U limit of the one-dimensional Hubbard model where (Penc *et al* 1995) wrote the electron creation operator as

$$\psi_\sigma^\dagger(0) = Z_{0,\sigma}^\dagger \Psi^\dagger(0), \qquad (14)$$

where $Z_{0,\sigma}^\dagger$ creates a site on the spin chain equation (12) with spin projection σ. The expression in equation (14) can be physically motivated as follows. From equation (13) it is clear that the creation of an electron is also accompanied by the creation of a holon. However, electrons also carry spin so there must be a component of the electron operator that also creates spin. This is accomplished by $Z_{0,\sigma}^\dagger$. In general, one has $Z_{l,\sigma}^\dagger$ as the object that adds a new site to the spin chain between $l - 1$ and l. While this appears physically intuitive, the expression suffers from the drawback that it does not naturally account for the variation of electron density with position in a real electron gas (Penc *et al* 1996). The remedy for this issue (Matveev *et al* 2007b) is to define the position at which the spin site is added to the chain equation (12) in terms of the number of holons to the left of the site:

$$l(x) = \int_{-\infty}^x \Psi^\dagger(y)\Psi(y)\,\mathrm{d}y. \qquad (15)$$

In terms of equation (15) the electron creation and annihilation operators are defined as

$$\psi_\sigma^\dagger(x) = Z_{l(x),\sigma}^\dagger \Psi^\dagger(x), \qquad (16)$$

$$\psi_\sigma(x) = \Psi(x) Z_{l(x),\sigma}. \qquad (17)$$

The operators given above explicitly account for the fact that the spins are attached to electrons and the formulae are valid at all energy scales. It is perhaps worth noting that, even though the Hamiltonian is spin–charge-separated, the electron operators are not written as a product of a spin piece and a charge piece because the 'spin' pieces $Z_{l(x),\sigma}$ also depend on the electron density via equation (15).

In writing equations (16) and (17) no assumptions have been made about the energy scale relative to the spin and charge energies. We now restrict our considerations to energies small compared to E_c, but arbitrary with respect to E_s. In this case, we are free to bosonize the holon sector. We have already given the form of the charge Hamiltonian in equation (11). The spinless fields θ and ϕ can be related to the holon density as (Fiete and Balents 2004, Fiete *et al* 2005b)

$$\Psi^\dagger(x)\Psi(x) = \frac{1}{\pi}[k_F^h + \partial_x \theta(x)], \qquad (18)$$

where the holon Fermi wavevector is twice the electron Fermi wavevector (Fiete and Balents 2004) $k_F^h = 2k_F$ and the bosonic fields satisfy the commutation relations $[\theta(x), \partial_y \phi(y)] = \mathrm{i}\pi\delta(x - y)$.

Since we are interested in low energies with respect to the charge energy, the electron operator may be expanded about the two-holon Fermi points at $\pm k_F^h$:

$$\Psi(x) = \Psi_R(x) + \Psi_L(x), \qquad (19)$$

where $\Psi_R(x)$ destroys a holon near the right Fermi point and $\Psi_L(x)$ destroys an electron near the left Fermi point. The left and right holon operators are bosonized as

$$\Psi_{R,L}(x) = \frac{1}{\sqrt{2\pi\alpha_c}} e^{-\mathrm{i}\phi(x)} e^{\pm \mathrm{i}[k_F^h + \theta(x)]}, \qquad (20)$$

where α_c is a short-distance cutoff of the order of the interparticle spacing a. Combining the results of equations (15) and (17)–(20) one obtains the bosonized form of the electron annihilation operator for spin σ:

$$\psi_\sigma(x) = \frac{e^{-\mathrm{i}\phi(x)}}{\sqrt{2\pi\alpha_c}}$$
$$\times \left(e^{\mathrm{i}[k_F^h x + \theta(x)]} + e^{-\mathrm{i}[k_F^h x + \theta(x)]} \right) Z_{l,\sigma} \Big|_{l=\frac{1}{\pi}[k_F^h x + \theta(x)]}, \qquad (21)$$

and an analogous expression for the electron creation operator $\psi_\sigma^\dagger(x)$. The expression equation (21), however, is not quite complete as it does not account for the discreteness of the charge of the electron. This can be accomplished by interpreting

$$Z_{l,\sigma}|_{l=\frac{1}{\pi}[k_F^h x + \theta(x)]} \rightarrow \sum_l Z_{l,\sigma} \delta\left(\frac{1}{\pi}[k_F^h x + \theta(x)] - l \right), \qquad (22)$$

after which the full electron annihilation operator (include both left- and right-moving parts) becomes (Matveev *et al* 2007b)

$$\psi_\sigma(x) = \frac{e^{-i\phi(x)}}{\sqrt{2\pi\alpha_c}} \int_{-\infty}^{\infty} \frac{dq}{2\pi} z_\sigma(q) e^{i(1+\frac{q}{\pi})[k_F^h x + \theta(x)]}, \quad (23)$$

where

$$z_\sigma(q) = \sum_{l=-\infty}^{\infty} Z_{l,\sigma} e^{-iql}. \quad (24)$$

The expression for the electron annihilation operator, equation (23), is the key result of (Matveev *et al* 2007b) who also showed that, in the limit of small energies compared to E_s, the expression correctly recovers the standard LL formulae for the electron annihilation operator. With equation (23) correlation functions involving electron operators can be expressed in terms of the correlation functions of the holon and spin sectors at arbitrary temperatures with respect to E_s, but small energies compared to E_c.

3.1.2. Tunneling density of states in an SILL. The tunneling density of states in an SILL is qualitatively different from the LL. While the tunneling density of states in an LL qualitatively resembles the behavior shown in figure 1(b) with $\omega_{th} = 0$, the tunneling density of an SILL qualitatively resembles the behavior in figure 1(a) with $\omega_{th} = 0$, at least for a range of frequencies, $k_B T \lesssim \hbar\omega \ll E_c$ and for not too strong interactions. For this reason, the tunneling density of states may provide the simplest and most transparent test for SILL physics in experiments.

Tunneling into an infinitely long SILL has been considered by (Cheianov and Zvonarev 2004a, Fiete and Balents 2004, Matveev *et al* 2007a). Deep in the spin-incoherent regime, the first-quantized path integral representation of the bosonized electron's Green's function is a convenient method (Fiete and Balents 2004, Fiete *et al* 2005b, Fiete 2007). Here we instead follow (Matveev *et al* 2007a, 2007b) as it provides a better launching point for numerical studies in the regime $k_B T \lesssim E_s \ll E_c$ and also readily captures some other important features of the tunneling density of states.

The tunneling density of states, $\nu(\omega) = -\frac{1}{\pi}\text{Im}[G^R(x = 0, \omega)]$, where $G^R(x = 0, \omega)$ is the Fourier transform of the retarded Green's function. The retarded Green's function generally contains particle and hole contributions, but for illustrative purposes, we will only focus on the particle contributions. Hole contributions are conceptually and technical similar, and in the spin-incoherent regime it can be shown that $\nu(\omega) = 2\nu(-\omega)$ for $\hbar\omega \gg k_B T$ (Matveev *et al* 2007a). Here we focus on the particle contribution: $G_\sigma^+(x, t) = \langle \psi_\sigma(x, t)\psi_\sigma^\dagger(0, 0)\rangle$. Using the general formulae equations (23) and (24) we have

$$G_\sigma^+(x, t) = \frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \sum_{l_1, l_2} e^{-i(q_1 l_1 - q_2 l_2)}$$

$$\times \langle Z_{l_1,\sigma} Z_{l_2,\sigma}^\dagger \rangle \langle e^{i[(1+\frac{q_1}{\pi})[k_F^h x + \theta(x,t)] - \phi(x,t)]}$$

$$\times e^{-i[(1+\frac{q_2}{\pi})\theta(0,0) - \phi(0,0)]}\rangle, \quad (25)$$

where we have assumed the time evolution of the spin degrees of freedom are slow compared to the charge degrees of freedom

so that we can neglect their time dependence (Matveev *et al* 2007b). In a translationally invariant system, one has $\langle Z_{l_1,\sigma} Z_{l_2,\sigma}^\dagger \rangle = \langle Z_{l_1-l_2,\sigma} Z_{0,\sigma}^\dagger \rangle$. Making a change of variables $l = l_1 - l_2$, the summation over the sites of the spin chain results in a delta function, $2\pi\delta(q_1 - q_2)$, which immediately kills one of the momentum integrals and sets $q_1 = q_2$. The resulting Green's function is

$$G_\sigma^+(x, t) = \frac{1}{2\pi\alpha_c} \int_{-\infty}^{\infty} \frac{dq}{2\pi} c_\sigma^+(q) e^{i(1+\frac{q}{\pi})k_F^h x} g_q^+(x, t), \quad (26)$$

where

$$c_\sigma^+(q) = \sum_{l=-\infty}^{\infty} \langle Z_{l,\sigma} Z_{0,\sigma}^\dagger \rangle e^{-iql}, \quad (27)$$

and

$$g_q^+(x, t) = \langle e^{i[(1+\frac{q}{\pi})\theta(x,t) - \phi(x,t)]} e^{-i[(1+\frac{q}{\pi})\theta(0,0) - \phi(0,0)]}\rangle. \quad (28)$$

Since the holon sector is described by the Gaussian theory, equation (11), we can make use of the identity $\langle e^{i\mathcal{O}}\rangle = e^{-\langle\mathcal{O}^2\rangle/2}$ for operator $\mathcal{O}$ and evaluate the correlation function equation (28) at zero temperature with respect to the charge energy. (Generalization to small but finite temperatures in the charge sector is trivial (Giamarchi 2004, Gogolin *et al* 1998).) The expression for the hole-like contributions to the Green's function has the same structure as equation (26) and is reported in (Matveev *et al* 2007b). Since for low temperatures compared to the charge energy the holon correlations in equation (28) can be analytically evaluated in a straightforward way, the emphasis for arbitrary temperatures with respect to the spin energy is on the spin correlations described by equation (27). For arbitrary temperatures the evaluation of $c_\sigma^+(q)$ is non-trivial (Matveev *et al* 2007b) and it is precisely at this point that numerical input for both zero and non-zero magnetic fields would be most helpful. Keeping for the moment temperature arbitrary with respect to E_s, but effectively zero compared to E_c, one finds for the tunneling density of states (Matveev *et al* 2007a)

$$\nu(\omega) = \nu_0 \int \frac{dq}{2\pi} \frac{c_\sigma^+(q)}{\Gamma(\lambda(q) + 1)} \left(\frac{\hbar\omega}{E_c}\right)^{\lambda(q)}, \quad (29)$$

where $\nu_0 = (\pi\hbar v_c)^{-1}$, $\lambda(q) = \frac{1}{2}[g(1 + \frac{q}{\pi})^2 + \frac{1}{g}] - 1$ and $g = 2g_c$ is the Luttinger interaction parameter appearing in equation (11). From the exponent $\lambda(q)$ one sees that the frequency dependence $\sim(\frac{\hbar\omega}{E_c})^{1/(2g)-1}e^{-\ln(E_c/\hbar\omega)(1+q/\pi)^2/2}$ which is maximum for $q = -\pi$. Expanding about this point and doing the Gaussian integration, one finds

$$\nu(\omega) \approx \nu_0 \frac{c_\sigma^+(-\pi)}{2\pi\Gamma(\lambda(-\pi) + 1)} \left(\frac{\hbar\omega}{E_c}\right)^{1/(2g)-1} \sqrt{\frac{2\pi}{g\ln(E_c/\hbar\omega)}}, \quad (30)$$

which agrees with the result obtained earlier for the infinite U limit of the Hubbard model (Cheianov and Zvonarev 2004a) and for the general interaction parameter g in the spin-incoherent regime (Fiete and Balents 2004). Note, however, that the result equation (30) is valid at $T = 0$. The only assumptions were that the dominant weight comes from the

point $q = -\pi$ and that $c_\sigma^+(-\pi) \neq 0$. If one is deep in the spin-incoherent regime $\langle Z_{l,\sigma} Z_{0,\sigma}^\dagger \rangle = (\frac{1}{2})^{|l|}$ *independent of the details and form* of H_s (a simple argument for this is given in (Fiete and Balents 2004, Fiete 2007)) and one has $c_\sigma^+(q) = \frac{3}{5-4\cos(q)}$. Hence, the assumptions leading to equation (30) are valid and the result indeed applies in the spin-incoherent regime as well. On the other hand, if the spins are fully polarized (along the z axis, say) then $c_\uparrow^+(q) = 2\pi\delta(q)$ and the tunneling density of states will necessarily have a different form (in fact, the familiar LL form) from equation (30) (Matveev *et al* 2007a). Also at $T = 0$ and zero magnetic field one has $c_\sigma^+(-\pi) = 0$ so the logarithmic factor appearing in equation (30) is absent and the LL form is obtained.

Finally, note that in the spin-incoherent regime where equation (30) applies, the tunneling response of an SILL looks qualitatively like the behavior in figure 1(a) with $\omega_{\rm th} = 0$ for $g = 2g_c > 1/2$, but it *does not* follow the analytical form of equation (1) because of the logarithmic corrections. If the interactions are both strong and long-ranged, it is possible to obtain $g_c < 1/4$, in which case the tunneling density of states would look qualitatively like the behavior in figure 1(b) with $\omega_{\rm th} = 0$ although it would still not fit the analytic form in equation (1). (Recall that for zero-range interactions the smallest value g_c can take is $1/2$, obtained in the infinite-U limit of the Hubbard model (Giamarchi 2004).) In this sense, the tunneling response of the SILL is in a different universality class from both the Fermi liquid and Luttinger liquid. We will now see that the same logarithmic corrections can also appear in the frequency dependence of the photon absorption.

3.2. Fermi-edge singularity in a spin-incoherent Luttinger liquid

As we discussed earlier, the physics of the Fermi-edge singularity falls into two classes: (1) electron backscattering from the hole is an irrelevant perturbation and (2) electron backscattering from the hole is a relevant perturbation. For a Luttinger liquid these two regimes are distinguished by whether the hole has a finite mass, case (1), or an infinite mass, case (2). In the SILL there is a subtlety because the relevance or irrelevance of backscattering from a local potential (for example, an infinite mass core hole) depends on the interactions in the system through the Luttinger parameter g. In particular, a local impurity is only a relevant perturbation if $g_c < 1/2$ (which is the same as $g < 1$) (Fiete *et al* 2005a). If $1/2 < g_c < 1$ the interactions of LL are repulsive, but a local impurity is an *irrelevant* perturbation. Therefore, in the SILL we effectively have three possible regimes for the Fermi-edge singularity: (i) finite mass mobile hole for which backscattering is irrelevant, (ii) infinite mass core hole for which backscattering is irrelevant and (iii) infinite mass core hole for which backscattering is relevant.

Let us first consider the case of a finite mass hole. In this case the backscattering is always an irrelevant perturbation. For addressing the threshold behavior of the photon absorption we can therefore neglect backscattering terms and focus only on forward scattering interactions between the electron and

the hole. In the initial study of the Fermi-edge singularity in an LL (Ogawa *et al* 1992) it was incorrectly concluded that the exponent α is independent of the dynamics of the hole, i.e. its mass. A more careful treatment showed that indeed the mass of the hole enters α (Tsukamoto *et al* 1998a, 1998b). The work of Tsukamoto *et al* employed a combined Bethe ansatz Conformal Field Theory study of a zero-range interaction between particles. The asymptotic structure of the energy and critical exponents in this model could be obtained exactly. From the formulae obtained it was shown explicitly that the exponent α depends on the mass of the hole. Perhaps the most important result is the structure of the electron–hole interaction: there are actually two forward scattering terms that emerge. One is the familiar density–density interaction between hole density and electron density. The other is less familiar and it describes the interaction of the hole with a current of electrons that is flowing by it. In other words, it is natural to transform to a frame co-moving with the hole (Castro Neto and Fisher 1996). In this frame the hole sees a current of electrons. One might think that such a picture would break time-reversal symmetry, but in an actual experiment electrons would be excited near both the right and left Fermi points shown in figure 2(a). Thus, as a whole, time-reversal symmetry is preserved. As the contribution to the absorption edge is the same in both cases, one need only focus on one Fermi point. Our discussion below will cover both Fermi points in one fell swoop, as the electron operator equation (23) contains both right-and left-moving pieces.

The Fermi-edge singularity in a spin-incoherent Luttinger liquid was discussed by (Fiete 2006). The full Hamiltonian for our problem in the frame co-moving with the hole is $H = H_{\rm elec} + H_{\rm elec-hole} + H_{\rm hole}$, where $H_{\rm elec}$ is given by equations (11) and (12), and

$$H_{\rm elec-hole} = \frac{U_s^{\rm f}}{\pi} h^\dagger h \partial_x \theta(0) \pm \frac{U_a^{\rm f}}{\pi} h^\dagger h \partial_x \phi(0), \qquad (31)$$

with $H_{\rm hole} = \sum_\sigma E_{{\rm h},\sigma} h_\sigma^\dagger h_\sigma$ and $h^\dagger h = \sum_\sigma h_\sigma^\dagger h_\sigma$. Here $U_s^{\rm f}$ is the symmetric part of the forward scattering from the hole and $U_a^{\rm f}$ is the antisymmetric part of the forward scattering (Tsukamoto *et al* 1998a). (In our convention $\partial_x \theta$ represents the density fluctuations and $\partial_x \phi$ the particle current.) Again, the antisymmetric part appears since, in the frame of the hole, it sees a net current of particles scattering from it. The '+' sign is for a right-moving hole and the '−' sign is for a left-moving hole. The parameter $U_a^{\rm f}$ depends on the momentum and mass of the hole, and when it is at rest, $U_a^{\rm f} \equiv 0$ (Tsukamoto *et al* 1998a). The operator $h_\sigma^\dagger$ creates a hole with spin σ and h_σ annihilates a hole with spin σ. The energy of such a hole in its rest frame is $E_{{\rm h},\sigma}$, which in the presence of an externally applied magnetic field can depend on σ.

To compute the photon absorption we treat the electron–photon interaction in second-order perturbation theory. Standard manipulations show (Mahan 1981)

$$I(\omega) \propto \sum_\sigma {\rm Re} \int_0^\infty {\rm d}t \, {\rm e}^{{\rm i}\omega t} \langle \psi_\sigma(t) h_\sigma(t) h_\sigma^\dagger(0) \psi_\sigma^\dagger(0) \rangle. \qquad (32)$$

To evaluate the correlations appearing in equation (32) we must diagonalize the Hamiltonian. The Hamiltonian H can

be diagonalized with the unitary transformation

$$U = \exp\{-i[\delta_a\theta(0) + \delta_s\phi(0)]h^\dagger h\} \qquad (33)$$

where

$$\delta_a \equiv \mp U_a^f/(vg\pi), \qquad (34)$$

and

$$\delta_s \equiv -gU_s^f/(v\pi). \qquad (35)$$

Applying this transformation we find $\bar{H} \equiv U^\dagger H U = H_{elec} + \bar{H}_{hole}$, where the only change to H_{hole} is a shift in the hole energy $E_{h,\sigma} \to \tilde{E}_{h,\sigma}$, which is unimportant to us here. Our main concern is with computing the threshold exponent α and the functional form of the response.

We begin with the evaluation of the correlation function $C_\sigma(\tau) = \langle\psi_\sigma(\tau)h_\sigma(\tau)h_\sigma^\dagger(0)\psi_\sigma^\dagger(0)\rangle$, where τ is the imaginary time. At finite temperature

$$C_\sigma(\tau) = \frac{1}{Z}\text{Tr}\left[e^{-\beta H}\psi_\sigma(\tau)h_\sigma(\tau)h_\sigma^\dagger(0)\psi_\sigma^\dagger(0)\right] \qquad (36)$$

$$= \frac{1}{Z}\text{Tr}[e^{-\beta\bar{H}}\bar{\psi}_\sigma(\tau)\bar{h}_\sigma(\tau)\bar{h}_\sigma^\dagger(0)\bar{\psi}_\sigma^\dagger(0)], \qquad (37)$$

where in the second line we have introduced the unitary transformation $U^\dagger U = 1$ and used the cyclic property of the trace. We have already determined $\bar{H}$, so it remains to determine $\bar{\psi}_\sigma$ and $\bar{h}_\sigma$. Direct evaluation gives $\bar{\psi}_\sigma = \psi_\sigma$ (up to unimportant multiplicative factors) and $\bar{h}_\sigma = h_\sigma e^{-i[\delta_a\theta + \delta_s\phi]}$. Therefore, the correlation function $C_\sigma(\tau)$ separates as $C_\sigma(\tau) = C_{\psi,\sigma}(\tau)C_{h,\sigma}(\tau)$. One readily finds $C_{h,\sigma}(\tau) = e^{-\tilde{E}_{h,\sigma}\tau}$ at zero temperature. This factor will enter the threshold frequency ω_{th} in equation (32) and will in general also depend on the external magnetic field. Our main interest here is in the evaluation of the part of the correlation function that will give us the frequency dependence just above threshold:

$$C_{\psi,\sigma}(\tau) = \frac{1}{Z_{elec}}\text{Tr}[e^{-\beta H_{elec}}\psi_\sigma(\tau)e^{-i[\delta_a\theta(\tau)+\delta_s\phi(\tau)]}$$
$$\times e^{i[\delta_a\theta(0)+\delta_s\phi(0)]}\psi_\sigma^\dagger(0)], \qquad (38)$$

where $Z_{elec} = \text{Tr}[e^{-\beta H_{elec}}]$. Formally, equation (38) bears a striking resemblance to the single-particle Green's function evaluated in the spin-incoherent regime above. It is worth pausing a moment to understand the physics of equation (38). Recalling that the operator $e^{i\phi(x,\tau)}$ creates a particle at spacetime point (x,τ) in the many-body system and the electron number is related to the θ field via $N(x,\tau) = \bar{n}x + \frac{1}{\pi}(\theta(x,\tau) - \theta(0,0))$, with $\bar{n}$ the average particle density, we see immediately that $C_{\psi,\sigma}(\tau)$ involves adding an electron *plus* an additional 'background excitation' (from $e^{i\delta_s\phi(0)}$) and then removing the same particle and its additional background at a time τ later. The factors $e^{\pm i\delta_a\theta}$ contribute additional density fluctuations coming from the motion of the finite mass valence hole. This factor is absent in the infinite mass limit.

It remains to evaluate the trace in equation (38). We again follow the more general bosonization scheme for strongly interacting systems described above which reduces to the previously obtained results deep in the spin-incoherent regime (Fiete 2006). We have

$$C_{\psi,\sigma}(\tau) = \frac{1}{2\pi\alpha_c}\int_{-\infty}^{\infty}\frac{dq}{2\pi}c_\sigma^+(q)f_q^+(\tau), \qquad (39)$$

where

$$f_q^+(\tau) = \langle e^{i[(1+\frac{q}{\pi}-\delta_a)\theta(\tau)-(1-\delta_s)\phi(\tau)]}e^{-i[(1+\frac{q}{\pi}-\delta_a)\theta(0)-(1-\delta_s)\phi(0)]}\rangle. \qquad (40)$$

In the zero-temperature limit with respect to the charge degrees of freedom, we can evaluate equation (40) using equation (11) as we did for the single-particle Green's function. The result is

$$f_q^+(\tau) = \left(\frac{\alpha_c}{v_c\tau + \alpha_c}\right)^{\frac{1}{2}[g(1+\frac{q}{\pi}-\delta_a)^2+\frac{1}{g}(1-\delta_s)^2]}. \qquad (41)$$

The frequency dependence of $I(\omega)$ can now be computed from equation (32) using $C_\sigma(\tau) = C_{\psi,\sigma}(\tau)C_{h,\sigma}(\tau)$. The computation is essentially identical to that of the tunneling density of states, equation (29), only $\omega \to \tilde{\omega} = \omega - \omega_{th}$ and $\lambda(q) \to \tilde{\lambda}(q) = \frac{1}{2}[g(1+\frac{q}{\pi}-\delta_a)^2+\frac{1}{g}(1-\delta_s)^2]-1$. Therefore, we have

$$I(\omega) \propto \sum_\sigma \int\frac{dq}{2\pi}\frac{c_\sigma^+(q)}{\Gamma(\tilde{\lambda}(q)+1)}\left(\frac{\hbar\tilde{\omega}}{E_c}\right)^{\tilde{\lambda}(q)}\Theta(\tilde{\omega}), \qquad (42)$$

where the $\Theta(\tilde{\omega} = \omega - \omega_{th})$ factor tells us there is no photon absorption below a threshold frequency ω_{th}. From figure 2(a) it is clear that $\hbar\omega_{th}$ is of order the bandgap plus the Fermi energy of the electrons plus the absolute value of the energy of the hole near the Fermi points of the electron gas. As with the expression equation (29) for the tunneling density of states, equation (42) is the general expression for the photon absorption in an interacting one-dimensional system. For temperatures low compared to the charge energy, but arbitrary with respect to the spin energy, the heavy lifting is in the evaluation of $c_\sigma^+(q)$, as we emphasized in the discussion of the single-particle Green's function. Following the same arguments we used there, we see that the frequency dependence $\sim(\frac{\hbar\tilde{\omega}}{E_c})^{\frac{1}{2g}(1-\delta_s)^2-1}e^{-\frac{g}{2}\ln[\frac{E_c}{\hbar\tilde{\omega}}](1+q/\pi-\delta_a)^2}$, which is a maximum for $q = \pi(\delta_a - 1)$. Expanding about this point and doing the Gaussian integration, one finds

$$I(\omega) \propto \sum_\sigma \frac{c_\sigma^+(\pi(\delta_a-1))}{2\pi\Gamma(\tilde{\lambda}(\pi(\delta_a-1))+1)}\left(\frac{\hbar\tilde{\omega}}{E_c}\right)^{\frac{1}{2g}(1-\delta_s)^2-1}$$
$$\times\sqrt{\frac{2\pi}{g\ln(\frac{E_c}{\hbar\tilde{\omega}})}}\Theta(\tilde{\omega}). \qquad (43)$$

Note that the final form of the photon absorption equation (43) is almost identical to that of the tunneling density of states, equation (30), aside from the factor δ_s that appears in the threshold exponent.

It is instructive to compare the similarities and differences of the Fermi-edge singularity with a mobile hole in the LL and SILL. Comparing with the general formula equation (1) we see that the threshold behavior is different in the SILL compared to the LL and even the Fermi liquid. The threshold behavior of the SILL contains the same logarithmic corrections that we found earlier in the tunneling density of states. Focusing on the power law part of the threshold behavior we see that

$$\alpha_{mobile}^{SILL} = 1 - \frac{1}{2g}(1-\delta_s)^2, \qquad (44)$$

so that it is independent of δ_a, unlike the LL case in equation (9). Since the asymmetric phase shift does not appear, this implies the threshold exponent is *independent of the hole dynamics and therefore the mass of the hole* in the spin-incoherent case. Of course, from equation (43) it is clear that the effects of δ_a are still felt, but only as a prefactor to the frequency dependence.

The effects of an externally applied magnetic field can also be easily incorporated (Fiete 2006). For a quantum wire the magnetic field does not affect the orbital part of the wavefunction until the magnetic length is of the order of the wire width. We therefore only discuss the coupling of the magnetic field to the spin of the electron, i.e. a Zeeman coupling. The magnetic field will influence the spin correlations described by $c_\sigma^+(q)$. In the spin-incoherent regime with zero field one has $\langle Z_{l,\sigma} Z_{0,\sigma}^\dagger \rangle = 2^{-|l|}$, independent of σ. In a finite external field one instead has $\langle Z_{l,\sigma} Z_{0,\sigma}^\dagger \rangle = p_\sigma^{|l|}$, where $p_\uparrow = (1 + \exp\{-E_Z/k_B T\})^{-1} = 1 - p_\downarrow$ and E_Z is the Zeeman energy (Fiete 2006, Kindermann and Brouwer 2006). While the presence of the magnetic field may shift the threshold energy, it will not affect the exponent equation (44) (Fiete 2006). The magnetic field also gives rise to a frequency scale $\omega^* = \frac{E_c}{\hbar} e^{-2E_Z/(gk_B T)}$ below which SILL behavior is observed and above which behavior characteristic of a spinless LL is observed (Fiete 2006, Kindermann and Brouwer 2006), namely the form equation (1) with the exponent equation (9).

We next turn to the case of an infinitely massive impurity for which backscattering is irrelevant. In this case the effective electron–hole interaction is given by equation (31) only with $\delta_a = 0$ as this corresponds to the case of a stationary hole with no backscattering. The photon absorption in this case is thus given by equation (43) with $\delta_a = 0$. While an infinite mass hole breaks translational symmetry, the irrelevance of backscattering effectively restores translational symmetry at low energies and the threshold behavior maps onto the finite mass case.

On the other hand, when backscattering is relevant the situation is very different. At low energies the hole acts as a boundary to a semi-infinite one-dimensional system (Kane and Fisher 1992, Furusaki and Nagaosa 1993). In this case, the SILL maps onto a *spinless* electron system (Fiete *et al* 2005a, Fiete 2007) which allows us to immediately exploit (Fiete 2006) the results obtained for that system. In particular, one obtains a universal contribution of $1/8$ to the threshold exponent (Kane *et al* 1994, Prokof'ev 1994, Affleck and Ludwig 1994, Furusaki 1997, Kominik *et al* 1997):

$$\alpha_{\text{immobile,relevant}}^{\text{SILL}} = 1 - \frac{1}{g}(1 - \delta_s)^2 - 1/8, \qquad (45)$$

and the frequency dependence follows equation (1) *without* the logarithmic corrections. In fact, for the infinitely massive hole with relevant backscattering the only signature of spin-incoherent physics is hidden in the mapping $g = 2g_c$ (Fiete *et al* 2005a).

4. Outstanding issues in the spin-incoherent Luttinger liquid

While our theoretical understanding of the spin-incoherent regime has advanced rapidly, the difficulty of obtaining the regime $E_s \ll E_c$ has hampered experimental tests of the theory. To date, our best experimental evidence comes from momentum-resolved tunneling on cleaved-edge overgrowth quantum wires (Steinberg *et al* 2006), transport in split-gate devices (Hew *et al* 2008), and somewhat more speculatively in transport measurements on gated single-wall carbon nanotubes with low electron density (Deshpande and Bockrath 2008). Given that the existing theory now encompasses hybrid structures of SILLs with ferromagnets and superconductors as well (Tilahun and Fiete 2008, 2009)), the number of experimental groups poised to make contributions to this exciting field is greatly enlarged. For a recent discussion of numerous possible experiments that would provide strong evidence for spin-incoherent Luttinger liquid behavior see (Fiete 2007).

On the theoretical side, a number of open issues remain. In particular, it is desirable to have a better understanding of the behavior on temperature scales $k_B T \approx E_s$, which is most likely very relevant for many experiments that approach but do not quite reach the spin-incoherent regime. Related to this is the need for a better understanding of the crossover between the Luttinger liquid and the spin-incoherent Luttinger liquid regimes. Both of these issues will likely require a numerical attack as there are no obvious analytical methods available to address them. Finally there is the issue of spin–orbit coupling that has so far received no attention. For very strong spin–orbit coupling, is there novel behavior in the regime $E_s \ll k_B T \ll E_{SO}, E_c$? These issues await further study.

Acknowledgments

References

Affleck I and Ludwig A W W 1994 *J. Phys. A: Math. Gen.* **27** 5375–92
Anderson P W 1967 *Phys. Rev. Lett.* **18** 1049–51
Balents L 2000 *Phys. Rev. B* **61** 4429
Berkovich A 1991 *J. Phys. A: Math. Gen.* **24** 1543
Calleja J M, Goni A R, Dennis B S, Weiner J S, Pinczuk A, Schmitt-Rink S, Pfeiffer L N, West K W, Muller J F and Ruckenstein A E 1991 *Solid State Commun.* **79** 911
Calleja J M, Goni A R, Pinczuk A, Weiner B S, Pfeiffer L N and West K W 1995 *Phys. Rev. B* **51** 4285

Castella H 1996 *Phys. Rev.* B **54** 17422
Castro Neto A H and Fisher M P A 1996 *Phys. Rev.* B **53** 9713
Cheianov V V and Zvonarev M B 2004a *Phys. Rev. Lett.* **92** 176401
Cheianov V V and Zvonarev M B 2004b *J. Phys. A: Math. Gen.* **37** 2261
Deshpande V and Bockrath M 2008 *Nat. Phys.* **4** 314
Fiete G A 2006 *Phys. Rev. Lett.* **97** 256403
Fiete G A 2007 *Rev. Mod. Phys.* **79** 801
Fiete G A and Balents L 2004 *Phys. Rev. Lett.* **93** 226401
Fiete G A, Le Hur K and Balents L 2005a *Phys. Rev.* B **72** 125416
Fiete G A, Qian J, Tserkovnyak Y and Halperin B I 2005b *Phys. Rev.* B **72** 045315
Fisher M P A and Glazman L I 1997 *Mesoscopic Electron Transport* ed L Sohn, L Kouwenhoven and G Schön (Dordrecht: Kluwer–Academic)
Fogler M and Pivovarov E 2005 *Phys. Rev.* B **72** 195344
Fritze M, Nurmikko A V and Hawrylak P 1993 *Phys. Rev.* B **48** 4960–3
Furusaki A 1997 *Phys. Rev.* B **56** 9352
Furusaki A and Nagaosa N 1993 *Phys. Rev.* B **47** 4631
Giamarchi T 2004 *Quantum Physics in One Dimension* (Oxford: Clarendon)
Gogolin A O, Nersesyan A A and Tsvelik A M 1998 *Bosonization and Strongly Correlated Systems* (Cambridge: Cambridge University Press)
Haldane F 1981 *J. Phys. C: Solid State Phys.* **14** 2585
Häusler W 1996 *Z. Phys.* B **99** 551
Hew W K, Thomas K J, Pepper M, Farrer I, Anderson D, Jones G A C and Ritchie D A 2008 *Phys. Rev. Lett.* **101** 036801
Ihara T, Hayamizu Y, Yoshita M, Akiyama H, Pfeiffer L N and West K W 2007 *Phys. Rev. Lett.* **99** 126803
Kakashvili P, Bhongale S G, Pu H and Bolech C J 2008 *Phys. Rev.* A **78** 041602
Kakashvili P and Johannesson H 2007 *Phys. Rev.* B **76** 085128
Kane C L and Fisher M P A 1992 *Phys. Rev. Lett.* **68** 1220

Kane C L, Matveev K A and Glazman L I 1994 *Phys. Rev.* B **49** 2253
Kindermann M and Brouwer P W 2006 *Phys. Rev.* B **74** 115121
Klironomos A D, Ramazashvili R R and Matveev K A 2005 *Phys. Rev.* B **72** 195343
Kominik A, Egger R and Gogolin A O 1997 *Phys. Rev.* B **56** 1153
Lee D K K and Chen Y 1992 *Phys. Rev. Lett.* **69** 1399–402
Mahan G D 1967 *Phys. Rev.* **163** 612–7
Mahan G D 1981 *Many-Particle Physics* (New York: Plenum)
Matveev K A 2004a *Phys. Rev. Lett.* **92** 106801
Matveev K A 2004b *Phys. Rev.* B **70** 245319
Matveev K A, Furusaki A and Glazman L I 2007a *Phys. Rev. Lett.* **98** 096403
Matveev K A, Furusaki A and Glazman L I 2007b *Phys. Rev.* B **76** 155440
Müller-Hartmann E, Ramakrishnan T V and Toulouse G 1971 *Phys. Rev.* B **3** 1102–19
Nozières P and de Dominicis C T 1969 *Phys. Rev.* **178** 1097–107
Ogawa T 2004 *J. Phys.: Condens. Matter* **16** S3567
Ogawa T, Furusaki A and Nagaosa N 1992 *Phys. Rev. Lett.* **68** 3638
Ohtaka K and Tanabe Y 1990 *Rev. Mod. Phys.* **62** 929–91
Otani H and Ogawa T 1996a *Phys. Rev.* B **53** 4684
Otani H and Ogawa T 1996b *Phys. Rev.* B **54** 4540
Penc K, Hallberg K, Mila F and Shiba H 1996 *Phys. Rev. Lett.* **77** 1390–3
Penc K, Mila F and Shiba H 1995 *Phys. Rev. Lett.* **75** 894
Prokof'ev N V 1994 *Phys. Rev.* B **49** 2148
Ruckenstein A E and Schmitt-Rink S 1987 *Phys. Rev.* B **35** 7551–7
Steinberg H, Auslaender O M, Yacoby A, Qian J, Fiete G A, Tserkovnyak Y, Halperin B I, Baldwin K W, Pfeiffer L N and West K W 2006 *Phys. Rev.* B **73** 113307
Tilahun D and Fiete G A 2008 *Phys. Rev.* B **77** 140505(R)
Tilahun D and Fiete G A 2009 *Phys. Rev.* B **79** 115425
Tsukamoto Y, Fujii T and Kawakami N 1998a *Phys. Rev.* B **58** 3633
Tsukamoto Y, Fujii T and Kawakami N 1998b *Eur. Phys. J.* B **5** 479
Voit J 1995 *Rep. Prog. Phys.* **58** 977